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HYDRODYNAMIC STABILITY BY VARIATIONAL METHODS

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Abstract—A general variational equation is proposed for the study of the linear stability of non-isothermal stationary fluid flows. It is shown that the Euler–Lagrange equations of the variational criterion are the Orr–Sommerfeld relations of the stability problem. The general theory is applied to two specific examples. Firstly, the isothermal Poiseuille flow between two infinite parallel planes is considered; by using the self-consistent technique introduced by Glansdorff and Prigogine, the critical Reynolds number is found to be of the same order of magnitude as the values obtained by other authors. As a second example, one studies the consequence of a temperature dependent viscosity on the stability of a plane Couette flow. It is shown that for certain values of the parameters, the flow becomes unstable. Our results are compared with those of Sukanek *et al.* who were, to our knowledge, the first to treat this problem.

NOMENCLATURE

a, constant in equation (5.3);
a_i, constant variational parameter;
$\mathbf{A}$, matrix defined by expression (4.8);
$\mathbf{A}^i$, matrix appearing in the characteristic equation (5.10) and defined in appendix;
b, constant in equation (5.3);
b_i, constant variational parameter;
$\mathbf{B}$, matrix defined by expression (4.9);
B_r, Brinkmann number;
c, stability parameter;
c_p, specific heat at constant pressure;
$\mathbf{C}$, matrix defined by expression (4.10);
$\mathbf{C}^i$, matrix defined in appendix;
d_j, constant variational parameter;
D, derivation operator defined as d/dX_2;
e, constant in equation (5.3);
e_i, constant variational parameter;
f_i, trial function;
f_v, Helmholtz free energy per unit volume;
F_i, ith component of the body force;
g, dimensionless viscosity;
h, half distance between the plates;
I, functional submitted to variation;
$\mathbf{I}$, unit matrix;
J_i, ith component of the heat flux vector;
k, kinetic energy per unit volume;
$\mathscr{L}$, Orr–Sommerfeld linear operator;
L, Lagrangian density;
M, phenomenological coefficient;
$\mathbf{M}^i$, matrix defined in appendix;
N, phenomenological coefficient;
$\mathbf{N}^i$, matrix defined in appendix;
$\mathbf{O}^i$, matrix defined in appendix;
p, pressure;
$\hat{p}$, perturbation pressure amplitude;
p_j, trial function;
P, dimensionless pressure;
$\mathbf{P}^i$, matrix defined in appendix;
P_e, Peclet number;
P_r, Prandtl number;
q_i, trial function;
R, Reynolds number;
$\mathbf{R}^i$, matrix defined in Appendix;
R_c, critical Reynolds number;
s_v, entropy per unit volume;
t, time;
T, temperature;
$\mathbf{T}^i$, matrix defined in appendix;
T_i, Chebyshev polynomial;
u_i, ith component of the dimensionless velocity vector;
$\hat{u}_i$, perturbation velocity amplitude;
u_v, internal energy per unit volume;
$u_v[T, \mu]$, Legendre transform of u_v with respect to T and μ;
U, stationary axial velocity component ($U = u_2$);
$\mathbf{U}^i$, matrix defined in appendix;
v_i, ith component of the velocity vector;
V_{ij}, rate of deformation tensor;
W, amplitude of the velocity perturbation $u'_2(W = \hat{u}'_2)$;
x_i, space coordinate (x_1, axial coordinate; x_2, vertical coordinate);
X_i, dimensionless space coordinate;
y_i, space coordinate;
Y, dimensionless space coordinate ($Y = X_2$).

Greek symbols

α, wave number;
α_c, critical wave number;
β, positive constant in equation (3.4);
δ, variational operator;
δ_{ij}, Kronecker symbol;
Δ, first order deviation with respect to the unperturbed state;
ζ, quantity equal to gDU;
η, shear viscosity;
θ, dimensionless temperature;
$\hat{\theta}$, temperature perturbation amplitude;
Θ, quantity equal to $i\alpha\theta$;
λ, heat conductivity;
λ_i, roots of equation (4.6);
λ_L, Lagrange multiplier;
μ, chemical potential;
π_{ij}, pressure tensor;
ρ, density;
σ, entropy production per unit time and volume;
τ, dimensionless time;
ϕ, general solution of the Orr–Sommerfeld equation;
χ, heat diffusivity;
ψ, energy dissipation potential;
Ω, volume.

Subscript

0, non-varied quantity during variation.

Superscript

*, reference quantity;
$'$, small deviation with respect to the stationary reference state;
T, transposed quantity.

1. INTRODUCTION

RECENTLY, there has been considerable interest in approaching the problem of hydrodynamic stability by variational methods [1–7].

Lee and Reynolds [1] consider the linear stability of plane parallel stationary flows in bounded domains; all the fluid parameters like density, heat conductivity and viscosity are taken constant while no temperature gradients were imposed on the boundaries. The temperature effects being neglected, the problem is completely determined by the Orr–Sommerfeld equation for the velocity disturbance. Let us write it in the general form

$$\mathscr{L}\phi = 0 \tag{1.1}$$

where $\mathscr{L}$ is the Orr–Sommerfeld linear operator and ϕ the velocity disturbance. Calling ϕ^* the solution of the adjoint problem

$$\mathscr{L}^*\phi^* = 0, \tag{1.2}$$

Lee and Reynolds use as variational equation

$$\delta I = \delta \int_{y_1}^{y_2} \phi^* \mathscr{L} \phi \, \mathrm{d}y = 0.$$

It is easy to verify that one recovers (1.1) and (1.2) as Euler–Lagrange equations. Lee and Reynolds have investigated the stability of the plane Couette, the plane Poiseuille and the jet flows. With relatively simple trial functions, they obtain very accurate results.

Glansdorff and Prigogine [2] have established a general variational criterion, based on the concept of local potential, to calculate the onset of instability in fluids in motion. More general than the theory of Lee and Reynolds, temperature effects are now taken into consideration. It is shown that the local potential is an extremum when the linearized Orr–Sommerfeld equations for the velocity and the temperature perturbations are satisfied. Glansdorff–Prigogine's theory has been widely and successfully applied by Platten [3, 4], Schechter, Prigogine and Hamm [5], Schechter and Himmelbau [6], Butler *et al.* [7].

Recently, Lebon and Lambermont [8, 9] generalized the classical principle of Hamilton to hydrodynamics. The purpose of this work is to extend their results to the problem of hydrodynamic stability.

In Section 2, a general variational principle for the stability problem is established. It is shown that the Euler–Lagrange equations are identical with the set of the first order perturbed conservation equations. Expanding, as is usually done, the disturbances in terms of normal modes, another expression for the variational criterion is derived whose Euler–Lagrange equations are the Orr–Sommerfeld relations. The particular expression of the criterion for plane parallel flows is given in Section 3.

As an illustration, the stability of the plane Poiseuille flow is investigated (Section 4). Choosing the functions of Chandrasekhar [10] and Reid [11] as trial functions, the critical Reynolds number and the critical wave number are calculated. They are found to be in good agreement with recent results obtained by Orszag [12] and Chock and Schechter [13].

As a further application, the Couette motion between parallel plates is treated in Section 5. For constant density, heat conductivity and viscosity, it is well known [14–17] that the flow is always stable at any value of the Reynolds number. However, if the fluid characteristics are made temperature dependent so that the velocity profile of the stationary flow presents a point of inflexion, instability may occur [18].

In particular, this condition is fulfilled when the viscosity is assumed to decrease exponentially with the temperature. The influence of non-constant viscosity on the stability of the temperature and velocity distributions has been studied by Joseph [19, 20] and more recently by Sukanek, Goldstein and Laurence [21]. Like them, an incompressible fluid with constant heat conductivity and temperature exponentially decreasing viscosity is considered. The eigenvalue problem is solved by means of the self-consistent technique while combinations of Chebyshev functions are used as trial functions. A comparison between our results and those of Sukanek *et al.* is given in Section 5.

2. A VARIATIONAL CRITERION FOR THE STUDY OF HYDRODYNAMIC STABILITY

2.1 *Lebon–Lambermont's variational criterion for hydrodynamics*

Let us briefly recall some definitions and introduce some notations.

According to the local equilibrium hypothesis [2], the Gibbs relation for a one-component fluid is given by

$$du_v = T\,ds_v + \mu\,d\rho; \tag{2.1}$$

u_v and s_v are respectively the internal energy and the entropy per unit volume, T is the Kelvin temperature, ρ the density and μ the chemical potential.

The Legendre transform of u_v with respect to T and μ is defined as

$$u_v[T,\mu] = u_v - Ts_v - \rho\mu. \tag{2.2}$$

In virtue of Euler's relation, this quantity is just the thermodynamic pressure, p. By differentiation, one gets

$$du_v[T,\mu] \equiv -dp = -s_v\,dT - \rho\,d\mu. \tag{2.3}$$

Let us also introduce a Lagrangian density

$$L = k - u_v[T,\mu], \tag{2.4}$$

where

$$k = \tfrac{1}{2}\rho v_i v_i \tag{2.5}$$

is the kinetic energy per unit volume and v_i the ith cartesian component of the velocity; from now on, the Einstein convention of summation on repeated indices will be used.

According to the theory of irreversible thermodynamics [2, 9], the energy dissipated per unit time and volume inside the viscous fluid is expressed by:

$$T\sigma = -J_i\left(\frac{1}{T}\frac{\partial T}{\partial x_i}\right) - T\tilde{\pi}_{ij}\left(\frac{1}{T}\tilde{V}_{ij}\right), \tag{2.6}$$

σ is the entropy per unit time and volume, J_i the ith component of the heat flux vector, $\tilde{V}_{ij}$ the deviatoric part of the rate of deformation tensor and $\tilde{\pi}_{ij}$ the deviatoric part of the pressure tensor; those latter quantities are defined by

$$V_{ij} = \tfrac{1}{3}V_{kk}\delta_{ij} + \tilde{V}_{ij}, \tag{2.7}$$

$$\pi_{ij} = p\delta_{ij} + \tilde{\pi}_{ij}, \tag{2.8}$$

with

$$p = \tfrac{1}{3}\pi_{ii}.$$

The phenomenological laws inferred from (2.6) are given by

$$J_i = -\frac{M(T)}{T}\frac{\partial T}{\partial x_i}, \tag{2.9}$$

$$\tilde{\pi}_{ij} = -\frac{N(T)}{T^2}\tilde{V}_{ij}; \tag{2.10}$$

the phenomenological coefficients M and N are related to the heat conductivity coefficient λ and the shear viscosity η respectively by

$$M(T) = T\lambda(T), \tag{2.11}$$

$$N(T) = T^2\eta(T). \tag{2.12}$$

Introducing the so-called energy dissipation potential

$$\psi = \tfrac{1}{2}T\sigma,$$

one obtains after substituting (2.9) and (2.10) in (2.6):

$$\psi = \frac{1}{2}\frac{M}{T^2}\left(\frac{\partial T}{\partial x_i}\right)^2 + \frac{1}{2}\frac{N}{T^2}\tilde{V}_{ij}\tilde{V}_{ij}. \tag{2.13}$$

Consider now a fluid flowing through a fixed volume Ω during a time interval $t_1 - t_2$, let F_i be the body force per unit volume acting on the particles of the fluid. Lebon and Lambermont have shown that the non-stationary flow obey the following variational equation:

$$\delta_t\int_{t_1}^{t_2}\int_\Omega \frac{\partial L}{\partial t}\,d\Omega\,dt + \delta_x\int_{t_1}^{t_2}\int_\Omega \frac{\partial}{\partial x_i}(Lv_i)\,d\Omega\,dt + \delta\int_{t_1}^{t_2}\int_\Omega\left(\psi - F_i v_i + k_0\frac{\partial v_i}{\partial x_i}\right)d\Omega\,dt = 0. \tag{2.14}$$

The variations are to be taken with respect to the independent intensive variables μ, v_i and T; δ is the variation symbol, where δ_t and δ_x mean that the derivatives relative to time and space respectively are to be kept fixed during the variational procedure. Similarly, the density $\rho = \rho(\mu, T)$ appearing in the expression (2.5) for k as well as k_0 appearing in the third integral are to be held fixed during variation. These quantities are supposed to correspond to the presumed but not determined exact solution μ_0, v_{i0} and T_0 and therefore, their variations are necessarily equal to zero. However, at the end of the procedure, all the quantities are unmasked and the subsidiary conditions

$$\mu_0 = \mu,\ v_{i0} = v_i \text{ and } T_0 = T$$

are used. With the help of these relations, it has been shown [9] that the Euler–Lagrange equations corresponding to arbitrary variation of μ, v_i and T are just the following equations of mass, impulse and energy balance:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x_i}(\rho v_i), \tag{2.15}$$

$$\frac{\partial}{\partial t}(\rho v_i) = F_i - \frac{\partial}{\partial x_j}(\pi_{ij} + \rho v_i v_j), \tag{2.15'}$$

$$\frac{\partial}{\partial t} s_v = -\frac{1}{T}\frac{\partial J_i}{\partial x_i} - \frac{1}{T}\pi_{ij}V_{ij} - \frac{\partial}{\partial x_i}(v_i s_v). \tag{2.15''}$$

It must be pointed out that expression (2.14) is only valid for prescribed values of the intensive variables μ, T and v_i at the boundary. For more general boundary conditions, the reader is referred to [9].

2.2 *A variational criterion for linear stability*

The stability of the stationary motion can be studied by modifying slightly the variational principle (2.14).

A distrubance of the state parameters μ, v_i and T in the stationary state gives rise to a change of the Lagrangian L. Expanding it in Taylor serie, one gets

$$L_{\text{perturbed}} - L_{\text{stationary}} = \Delta L + \frac{1}{2!}\Delta^2 L + \dots$$

where the symbol Δ represents the first order deviation with respect to the reference stationary state; it must not be confused with the variation operator δ.

In virtue of the definition (2.4) of L, one has

$$\Delta L = \Delta k - \Delta u_v(T, \mu), \tag{2.16}$$

$$\Delta^2 L = \Delta^2 k - \Delta^2 u_v(T, \mu). \tag{2.17}$$

Representing by an upper prime a disturbed quantity of the first order,

$$g' = g_{\text{pert.}} - g_{\text{stat.}} \tag{2.18}$$

it is easy to verify that:

$$\Delta k = \rho v_i v_i' + \tfrac{1}{2} v_i^2 \rho', \tag{2.19}$$

$$\Delta^2 k = \rho (v_i')^2 + 2 v_i \rho' v_i' \tag{2.20}$$

and

$$\Delta u_v(T, \mu) = -s_v T' - \rho \mu', \tag{2.21}$$

$$\Delta^2 u_v(T, \mu) = -s_v' T' - \rho' \mu'. \tag{2.22}$$

We also need the explicit expression of $\Delta^2 \psi$. From (2.13) we obtain

$$\Delta^2 \psi = \frac{1}{T^2}\left(\frac{\partial T'}{\partial x_i} - \frac{T'}{T}\frac{\partial T}{\partial x_i}\right)\left(2\frac{\partial T}{\partial x_i}M' + M\frac{\partial T'}{\partial x_i} - 3M\frac{T'}{T}\frac{\partial T}{\partial x_i}\right) + \frac{1}{T^2}\left(\tilde{V}_{ij}' - \frac{T'}{T}\tilde{V}_{ij}\right) \times \left(2\tilde{V}_{ij}N' + N\tilde{V}_{ij}' - 3N\frac{T'}{T}\tilde{V}_{ij}\right). \tag{2.23}$$

We are now in position to formulate the variational criterion describing the perturbed motion. It states that

$$\delta_t \int_{t_1}^{t_2}\int_\Omega \frac{\partial}{\partial t}(\Delta^2 L)\,dt\,d\Omega + \delta_x \int_{t_1}^{t_2}\int_\Omega \frac{\partial}{\partial x_i}[\Delta^2(L v_i)]\,dt\,d\Omega + \delta \int_{t_1}^{t_2}\int_\Omega \left(\Delta^2\psi - F_i' v_i' + k_0'\frac{\partial v_i'}{\partial x_i}\right) dt\,d\Omega = 0, \tag{2.24}$$

together with the subsidiary conditions

$$\mu' = \mu_0', \; v_i' = v_{i0}', \; T' = T_0' \tag{2.25}$$

where index 0 refers to the exact perturbed solution. The quantities to be varied independently are μ', v_i' and T'. The Euler–Lagrange equations corresponding to arbitrary variations of these parameters restitute the first order disturbed balance equations of mass, impulse and energy, i.e.

$$\frac{\partial \rho'}{\partial t} = -\rho\frac{\partial v_i'}{\partial x_i} - \rho'\frac{\partial v_i}{\partial x_i}, \tag{2.26}$$

$$\rho\frac{\partial v_i'}{\partial t} + v_i\frac{\partial \rho'}{\partial t} = F_i' - \frac{\partial}{\partial x_i}(\pi_{ij}' + \rho' v_i v_j + \rho v_i' v_j + \rho v_i v_j'), \tag{2.27}$$

$$\frac{\partial}{\partial t}s_v' = -\frac{1}{T}\frac{\partial J_i'}{\partial x_i} + \frac{T'}{T^2}\frac{\partial J_i}{\partial x_i} - \frac{1}{T}\pi_{ij}'V_{ij} - \frac{1}{T}\pi_{ij}V_{ij}' + \frac{T'}{T^2}\pi_{ij}V_{ij} - \frac{\partial}{\partial x_i}(v_i s_v' + s_v v_i'). \tag{2.28}$$

Moreover, when the phenomenological coefficients M and N are temperature dependent, their disturbance must be kept constant during variation, i.e.

$$\delta M' = \delta N' = 0. \tag{2.29}$$

It must also be pointed out that the criterion (2.24) has been formulated for fixed values of the disturbances μ', v_i' and T' at the boundary; in fact, this is the most frequent situation met in practice.

If the *fluid is incompressible*, the density ρ is constant and the Gibbs equation reduces to:

$$du_v = T\,ds_v. \tag{2.30}$$

The Legendre transform with respect to T

$$u_v(T) = u_v - T s_v \equiv f_v \tag{2.31}$$

is called the Helmholtz free energy per unit volume f_v. The Lagrangian density L is now given by

$$L = k - f_v. \tag{2.32}$$

Another consequence of incompressibility is that the components of the velocity disturbance are no longer independent but linked by

$$\frac{\partial}{\partial x_i}v_i' = 0. \tag{2.33}$$

Therefore, expression (2.24) of the variational principle has to be modified in the form:

$$\delta_t \int_{t_1}^{t_2} \int_\Omega \frac{\partial}{\partial t}(\Delta^2 L)\, d\Omega\, dt + \delta_x \int_{t_1}^{t_2} \int_\Omega \frac{\partial}{\partial x_i}[\Delta^2 L v_i)]\, d\Omega\, dt$$
$$+ \delta \int_{t_1}^{t_2} \int_\Omega \left(\Delta^2 \psi - F_i' v_i' + \lambda_L(x_i, t) \frac{\partial v_i'}{\partial x_i}\right) d\Omega\, dt = 0, \quad (2.34)$$

where $\lambda_L(x_i, t)$ is a Lagrange multiplier.

3. STABILITY OF FLUID FLOWS BETWEEN PARALLEL PLANES

3.1 *The perturbed equations of motion*

Consider a Newtonian liquid in motion in the direction x_1 between two infinite planes normal to the x_2 direction and separated by a distance $2h$. The origin of the coordinate system is located halfway between the two plates. The density and the specific heat of the fluid are assumed constant.

In terms of the heat conductivity $\lambda(T)$ and the shear viscosity $\eta(T)$, the phenomenological equations (2.9) and (2.10) can be written as follows:

$$J_i = -\lambda(T) \frac{\partial T}{\partial x_i} \text{ (Fourier's law)}, \quad (3.1)$$

$$\tilde{\pi}_{ij} = -\frac{\eta(T)}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \text{ (Newton–Stokes' law)}. \quad (3.2)$$

As most liquids are characterized by a constant heat conductivity and an exponential dependence of the viscosity with respect to the temperature, we shall take:

$$\lambda = \text{constant}, \quad (3.3)$$

$$\eta = \eta^* \exp\left(\frac{\beta(T - T^*)}{T^*}\right), \quad (3.4)$$

β is a positive constant and η^* is the viscosity at the reference temperature T^*.

As in Section 2, let p, T, v_i be the solutions of the basic stationary flow and p', T', v_i' be small deviations from this state. For convenience, we introduce the dimensionless quantities

$$X_i = \frac{x_i}{h}, \quad \tau = \frac{v^* t}{h}, \quad u_i = \frac{v_i}{v^*},$$

$$\theta = \beta \frac{T}{T^*}, \quad P = \frac{p'}{\rho v^2}, \quad g = \frac{\eta}{\eta^*} \quad (3.5)$$

and the characteristic numbers

$$B_r = \beta \frac{\eta^* v^{*2}}{\lambda T^*} \text{ (Brinkmann's number)},$$

$$R = \frac{\rho v^* h}{\eta^*} \text{ (Reynolds' number)},$$

$$P_e = \frac{\rho c_p v^* h}{\lambda} \text{ (Peclet's number)}, \quad (3.6)$$

v^* is a reference velocity, for instance the relative velocity of the plates in the Couette flow and c_p the specific heat at constant pressure.

The linearized expressions of the perturbed mass, momentum and energy balance equations are respectively:

$$\frac{\partial}{\partial X_i} u_i' = 0, \quad (3.7)$$

$$\frac{\partial u_i'}{\partial \tau} + u_j \frac{\partial u_i'}{\partial X_j} + u_j' \frac{\partial u_i}{\partial X_j} = -\frac{\partial p'}{\partial X_i}$$
$$+ \frac{1}{R}\left\{\frac{\partial}{\partial X_j}\left(g\left(\frac{\partial u_i'}{\partial X_j} + \frac{\partial u_j'}{\partial X_i}\right) - \theta'\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)\right)\right\}, \quad (3.8)$$

$$\frac{\partial \theta'}{\partial \tau} + u_j \frac{\partial \theta'}{\partial X_j} + u_j' \frac{\partial \theta}{\partial X_j} = \frac{1}{P_e} \frac{\partial}{\partial X_j}\left(\frac{\partial \theta'}{\partial X_j}\right)$$
$$+ \frac{B_r}{P_e} g\left\{\left(\frac{\partial u_i'}{\partial X_j} + \frac{\partial u_j'}{\partial X_i}\right)\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)\right.$$
$$\left. - \theta'\left(\frac{\partial u_i}{\partial X_i} + \frac{\partial u_j}{\partial X_i}\right)\frac{\partial u_i}{\partial X_j}\right\}. \quad (3.9)$$

In the applications treated in this paper, the solutions of the basic stationary flow are of the form

$$u_1 = u_1(X_2) \equiv U, \; u_2 = 0 \text{ and } \theta = \theta(X_2). \quad (3.10)$$

3.2 *The variational equation for the perturbed equations of motion*

When temperature effects are neglected, Squire [22] has shown that, for incompressible and isoviscous fluids, it is justified to limit the stability analysis to two-dimensional disturbances. This result was extended to compressible but still isoviscous fluids by Lin [23]. In the case of a variable viscosity, Sukanek *et al.* [21] have proved that Squire's theorem is valid when applied to the momentum equation but fails for the energy equation. Consequently, a correct analysis should be conducted by considering three dimensional perturbations. However, it appears from the works of Goldstein [24] and Sukanek *et al.* [21] that, at least for the Couette flow, the modes associated with the energy equation are the most stable. Therefore and also because the greater numerical complexity of the three dimensional problem, we shall restrict ourselves to the two dimensional case. It follows that the disturbances ρ', u_i' and θ' will be of the form

$$f' = f'(X_1, X_2, \tau) \quad (3.11)$$

with $u_3' = 0$.

Prescribing the temperature and the velocity disturbances at the boundary, expression (2.34) of the variational principle becomes:

$$
\begin{aligned}
\delta I \equiv \delta \int_\tau \int_\Omega \bigg(& \frac{\partial u'_{01}}{\partial \tau} u'_1 + \frac{\partial u'_{02}}{\partial \tau} u'_2 + \frac{\partial U}{\partial X_2} u'_{02} u'_1 \\
& + U\left(\frac{\partial u'_{01}}{\partial X_1} u'_1 + \frac{\partial u'_{02}}{\partial X_1} u'_2\right) + \lambda_L\left(\frac{\partial u'_1}{\partial X_1} + \frac{\partial u'_2}{\partial X_2}\right) \\
& + \frac{P_e}{RB_r}\left(\frac{1}{\theta}\frac{\partial \theta'_0}{\partial \tau}\theta' + \frac{u'_{02}}{\theta}\frac{\partial \theta}{\partial X_2}\theta' + U\frac{\partial \theta'_0}{\partial X_1}\frac{\theta'}{\theta}\right) \\
& + \frac{1}{R}\left\{\frac{3}{2}\frac{g}{\theta^2}\left(\frac{\partial U}{\partial X_2}\right)^2 \theta'^2 - 2\frac{g}{\theta}\frac{\partial U}{\partial X_2}\left(\frac{\partial u'_1}{\partial X_2} + \frac{\partial u'_2}{\partial X_1}\right)\theta'\right. \\
& + g\left(\left(\frac{\partial u'_1}{\partial X_1} + \frac{\partial u'_2}{\partial X_2}\right)^2 + \frac{1}{2}\left(\frac{\partial u'_1}{\partial X_2} + \frac{\partial u'_2}{\partial X_1}\right)^2\right) + g\theta'_0\left(\frac{2}{\theta} - 1\right) \\
& \left. \times \left(\frac{\partial U}{\partial X_2}\left(\frac{\partial u'_1}{\partial X_2} + \frac{\partial u'_2}{\partial X_1}\right) - \frac{\theta'}{\theta}\left(\frac{\partial U}{\partial X_2}\right)^2\right)\right\} + \frac{1}{\theta R B_r} \\
& \times \left\{\frac{1}{2}\left(\left(\frac{\partial \theta'}{\partial X_1}\right)^2 + \left(\frac{\partial \theta'}{\partial X_2}\right)^2\right) + \frac{1}{\theta}\frac{\partial \theta}{\partial X_2}\frac{\partial \theta'}{\partial X_2}(\theta'_0 - 2\theta')\right. \\
& \left. + \frac{\theta'}{\theta^2}\left(\frac{\partial \theta}{\partial X_2}\right)^2\left(\frac{3}{2}\theta' - \theta'_0\right)\right\}\bigg) \mathrm{d}\tau\, \mathrm{d}\Omega = 0; \qquad (3.12)
\end{aligned}
$$

in this expression, we have for simplicity omitted all the terms giving a zero contribution to the Euler–Lagrange equations. It can be seen that the latter are yields the perturbed balance equations, it is more convenient to build up a principle which gives the Orr–Sommerfeld relations. This procedure reduces the number of independent variables from three to two, for instance u'_2 and θ'. It rests on the expansion of the quantities $u'_i (i = 1, 2)$, θ' and p' in the following form:

$$u'_i = \hat{u}_i(X_2)\exp(i\alpha(X_1 - c\tau)), \qquad (3.14)$$

$$\theta' = \hat{\theta}(X_2)\exp(i\alpha(X_1 - c\tau)), \qquad (3.15)$$

$$p' = \hat{p}(X_2)\exp(i\alpha(X_1 - c\tau)). \qquad (3.16)$$

According to the mass balance equation (3.7), the amplitudes $\hat{u}_1$ and $\hat{u}_2$ are related by:

$$\hat{u}_1 = -\frac{i}{\alpha}\frac{\mathrm{d}\hat{u}_2}{\mathrm{d}X_2}, \qquad (3.17)$$

or, in a more convenient form.

$$\hat{u}_1 = \frac{c}{\alpha} DW \text{ where } D = \frac{\mathrm{d}}{\mathrm{d}X_2} \text{ and } W = \hat{u}_2. \qquad (3.18)$$

Using (3.13) and substituting (3.14) to (3.16) in (3.12) gives an expression depending on $\hat{u}_1$, $\hat{u}_2$, $\hat{p}$ and $\hat{\theta}$. Eliminating $\hat{p}$ and $\hat{u}_1$ by means of equations (3.8) and (3.17) and introducing the new set of variables

$$\Theta = i\alpha\hat{\theta},\ \zeta = gDU \text{ and } X_2 = Y \qquad (3.19)$$

yields finally the following expression for (3.12):

$$
\begin{aligned}
\delta I \equiv \delta \int_{-1}^{+1} \bigg(& i\alpha R[(U-c)(\alpha^2 W_0 W + DWDW_0) - W_0 DUDW] - \frac{P_e}{B_r}\frac{i\alpha\Theta}{\theta}((U-c)\Theta_0 + W_0 D\theta) \\
& - \frac{1}{\theta B_r}\left\{\frac{1}{2}[(D\Theta)^2 + \alpha^2\Theta^2] + \frac{D\theta}{\theta}\Theta_0 D\Theta - \frac{2}{\theta}\Theta D\Theta D\theta - \frac{(D\theta)^2}{\theta^2}\Theta\left(\Theta_0 - \frac{3}{2}\Theta\right)\right\} \\
& + g\left(\frac{1}{2}\alpha^4 W^2 + \alpha^2 W(2D^2 W_0 - D^2 W) + 4\alpha^2 DWDW_0 - \frac{1}{2}(D^2 W)^2\right) - 2DWD[g(D^2 W_0 + \alpha^2 W_0)] \\
& - \zeta\left\{2D\Theta_0 DW + \frac{3}{2}\frac{DU}{\theta^2}\Theta^2 + \frac{2}{\theta}[\Theta(D^2 W + \alpha^2 W_0) - \alpha^2 W\Theta_0] + \left(1 - \frac{2}{\theta}\right)\Theta_0\left(D^2 W - \alpha^2 W + \frac{\Theta}{\theta}DU\right)\right\} \\
& + 2WD(\Theta_0 D\zeta)\bigg)\mathrm{d}Y = 0. \qquad (3.20)
\end{aligned}
$$

identical with the equations (3.8) and (3.9) when the subsidiary conditions

$$u'_{01} = u'_1,\ u'_{02} = u'_2,\ \theta_0 = \theta'_0$$

are introduced. The continuity equation (3.7) is not obtained but is used as a subsidiary condition to determine the value of λ_L, it is found that

$$\lambda_L = -p'. \qquad (3.13)$$

It must be observed that if it is desired to obtain the continuity equation as an Euler–Lagrange equation, it suffices to vary (3.12) with respect to λ_L.

3.3 *The variational equation for the Orr–Sommerfeld relations*

Instead of constructing a variational criterion which

After making use of the subsidiary conditions:

$$\Theta_0 = \Theta,\ W_0 = W$$

and taking the variations with respect to W and Θ respectively, one obtains the following Euler–Lagrange equations:

$$
\begin{aligned}
& D^2(gD^2 W) - 2\alpha^2 D(gDW) + \alpha^2(D^2 g)W \\
& \quad + \alpha^4 gW + i\alpha R(D^2 U)W - i\alpha R(U-c)(D^2 - \alpha^2)W \\
& \quad + (D^2 + \alpha^2)(\Theta\zeta) = 0, \qquad (3.21)
\end{aligned}
$$

$$
\begin{aligned}
& (D^2 - \alpha^2)\Theta - \zeta B_r \Theta DU - i\alpha P_e(U-c)\Theta \\
& \quad - i\alpha P_e W D\theta - 2\zeta B_r(D^2 + \alpha^2)W = 0 \qquad (3.22)
\end{aligned}
$$

which are the Orr–Sommerfeld equations of the problem.

The complete solution is obtained by adding the boundary conditions:

$$W = DW = \Theta = 0 \quad \text{at} \quad Y = \pm 1. \tag{3.23}$$

3.4 *Formulation of the eigenvalue problem*

To obtain an approximate solution of the stability problem, the self-consistent method proposed by Glansdorff and Prigogine [2] will be applied. This procedure is a generalization of Rayleigh–Ritz's technique. Like this one, the self-consistent method is closely related to the Galerkin's method [25, 26] and is even equivalent when the trial functions are linear expressions of the form

$$\phi = \sum_{i=1}^{n} a_i f_i,$$

where the a_i's are unknown constants while the f_i's are given a priori and satisfy the boundary conditions.

Returning to the Euler–Lagrange equations (3.21) and (3.22), let us assume that their solutions are approximated by

$$W = \sum_{i=1}^{n} a_i f_i(Y), \tag{3.24}$$

$$W_0 = \sum_{i=1}^{n} a_{i0} f_i(Y), \tag{3.24'}$$

$$\Theta = \sum_{j=1}^{m} d_j p_j(Y), \tag{3.25}$$

$$\Theta_0 = \sum_{j=1}^{m} d_{j0} p_j(Y). \tag{3.25'}$$

According to the self-consistent technique, W and W_0 (respectively Θ and Θ_0) are taken to have the same dependence with respect to the independent variable Y; the functions f_i and p_i constitute a complete set of functions obeying the boundary conditions of the problem, i.e.

$$f_i = Df_i = 0 \text{ and } p_j = 0 \text{ at } Y = \pm 1, \tag{3.26}$$

the a_i's and the d_j's are the unknown constant coefficients. After substituting the trial functions in expression (3.20) of the functional I, the a_i's and b_i's are determined by the stationary conditions:

$$\frac{\partial I(a_i, a_{i0}, d_j, d_{j0})}{\partial a_i} = 0 \quad i = 1, \ldots, n, \tag{3.27}$$

$$\frac{\partial I(a_i, a_{i0}, d_j, d_{j0})}{\partial d_j} = 0 \quad j = 1, \ldots, n. \tag{3.28}$$

After that all the derivations have been performed, index 0 is dropped and one is left with a system of $n+m$ homogeneous equations which can symbolically be written as:

$$(\mathbf{A} - c\mathbf{I}) \cdot \begin{pmatrix} \mathbf{a} \\ \mathbf{d} \end{pmatrix} = 0 \tag{3.29}$$

$\mathbf{A}$ is a matrix whose elements are made up of combination of integrals of f_i and p_j, $\mathbf{I}$ is the identity matrix while $\mathbf{a}$ and $\mathbf{d}$ are vectors with components $a_1, \ldots, a_n$ and $d_1, \ldots, d_m$ respectively. Non trivial solutions of (3.29) exist if and only if the secular equation

$$\det(\mathbf{A} - c\mathbf{I}) = 0,$$

so that we are faced with an eigenvalue problem. According to (3.14)–(3.16), the flow is unstable if the imaginary part, c_i, of c is positive. In the next section, we apply the foregoing analysis to two specific examples: the isothermal plane Poiseuille flow and the plane Couette flow with variable viscosity.

4. THE ISOTHERMAL POISEUILLE FLOW

In order to test our numerical methods, we first treat the problem of the stability of a Poiseuille flow between two infinite parallel planes at the same temperature. All the fluid characteristics like density, heat conductivity and viscosity are supposed constant. Moreover, the temperature perturbations are assumed to be negligible.

Under these restrictions, the functional (3.20) reduces to:

$$I = \int_{-1}^{+1} \Big\{ i\alpha R[(U-c)(\alpha^2 W_0 W + DWDW_0) - W_0 DUDW] + \frac{1}{2}\alpha^4 W^2 + \alpha^2 W(2D^2 W_0 - D^2 W) + 4\alpha^2 DWDW_0 - \frac{1}{2}(D^2 W)^2 - 2DWD(D^2 W_0 + \alpha^2 W_0) \Big\} \mathrm{d}Y \tag{4.1}$$

with

$$U = 1 - Y^2. \tag{4.2}$$

The Euler–Lagrange equation resulting from arbitrary variations of W is the well-known Orr–Sommerfeld relation

$$(D^2 - \alpha^2)^2 W - i\alpha R[(U-c)(D^2 - \alpha^2)W - WD^2 U] = 0. \tag{4.3}$$

Let us expand W and W_0 in the form

$$W = \sum_{i=1}^{n} a_i f_i(Y) \quad W_0 = \sum_{i=1}^{n} a_{i0} f_i(Y) \tag{4.4}$$

where the f_i's are orthogonal functions used earlier by Reid and Harris [11] and Chandrasekhar [10]. The f_i's are solution of the eigenvalue problem:

$$(D^4 + \lambda^4) f_i = 0 \quad f_i(\pm 1) = Df_i(\pm 1) = 0$$

and are given by

$$f_i = \frac{\cosh(\lambda_i Y)}{\cosh \lambda_i} - \frac{\cos(\lambda_i Y)}{\cos \lambda_i} \tag{4.5}$$

where the λ_i's are the roots of

$$\tanh \lambda_i + \tan \lambda_i = 0. \tag{4.6}$$

The orthogonality conditions satisfied by the f_i's are

$$\int_{-1}^{+1} f_i(Y) f_j(Y)\,\mathrm{d}Y = 2\delta_{ij}. \tag{4.7}$$

After rather long and fastidious but elementary manipulations, one obtains the following secular equation:

$$\mathbf{det}\left[\mathbf{C}^{-1}\left(\mathbf{B} - \frac{i\mathbf{A}}{\alpha R}\right) - c\mathbf{I}\right] = 0$$

where **A**, **B** and **C** are square matrices whose elements are given by:

$$A_{ij} = \alpha^4 F_{ij}^{00} + 2\alpha^2 F_{ij}^{11} + F_{ij}^{12} + \alpha^2(F_{ij}^{20} - F_{ji}^{20}) \tag{4.8}$$

$$B_{ij} = \alpha^2 V_{ij}^{00} + V_{ij}^{11} - D_{ij}^{10} \tag{4.9}$$

$$C_{ij} = \alpha^2 F_{ij}^{00} + F_{ij}^{11} \tag{4.10}$$

with

$$F_{ij}^{nm} = \int_{-1}^{+1} f_i^n f_j^m \,\mathrm{d}Y \qquad D_{ij}^{nm} = \int_{-1}^{+1} DU f_i^n f_j^m \,\mathrm{d}Y$$

$$V_{ij}^{nm} = \int_{-1}^{+1} U f_i^n f_j^m \,\mathrm{d}Y \quad n, m = 0, 1, 2$$

where f_i^n stands for $\mathrm{d}^n f_i/\mathrm{d}Y^n$ $(n = 1, 2)$ while f_i^0 represents the function f_i itself.

Integrals are determined numerically by the Gauss procedure. The inverse of the matrix **C** is evaluated by means of the Gauss–Jordan technique while the eigenvalues are obtained by using the Q.R. algorithm. All the computations are performed on the IBM 370 computer of the University of Liège.

The imaginary part of the eigenvalues has been calculated for $0{\cdot}5 < \alpha < 1{\cdot}5$ and $100 < R < 50\,000$. The number of terms in the expansion (4.4) has been limited to $n = 17$. In Table 1 are reported the results corresponding to $R = 250$ and $R = 3000$ with in both cases $\alpha = 1$.

Table 1. Convergence of the imaginary part c_i of the stability parameter

$\alpha = 1, R = 250$		$\alpha = 1, R = 3000$	
n	c_i	n	c_i
2	−0·084652	2	−0·007130
3	−0·134329	3	−0·014451
4	−0·101212	4	−0·027913
5	−0·102684	5	−0·065203
6	−0·103787	6	+0·008008
7	−0·104066	7	+0·008164
8	−0·104153	8	+0·003589
9	−0·104190	9	−0·002149
10	−0·104209	10	−0·006694
11	−0·104228	11	−0·009211
12	−0·104223	12	−0·010085
13	−0·104230	13	−0·010222
14	−0·104230	14	−0·010272
15	−0·104227	15	−0·010287
16	−0·104244	16	−0·010420
17	−0·104295	17	−0·010364

The convergence is satisfactory as long as R does not exceed 6000. For higher Reynolds numbers, the results become less accurate.

Some particular values of R and c_r giving neutral stability as a function of α are reproduced in Table 2; for comparison, the results obtained by Chock and Schechter [13] using the Runge–Kutta technique are also listed. It is seen that the results are very similar.

It is found that the critical Reynolds number R_c and the critical wave number α_c are respectively given by

$$R_c = 5727{\cdot}2 \qquad \alpha_c = 1{\cdot}02.$$

Table 2. Values of R and c_r corresponding to neutral stability as a function of the wave number α and comparison with the results of Chock and Schechter [13]

α	R		c_r	
	Lebon–Nguyen	Chock–Schechter	Lebon–Nguyen	Chock–Schechter
0·9	6969	6965·25	0·240246	0·240812
1	5770	5814·83	0·261066	0·261233
1·02	5727·2	5772·26	0·263465	0·263936
1·021	5727·5	5772·25	0·26388	0·264053
1·022	5728	5775·47	0·26400	0·264168
1·026	5732	5775·61	0·264414	0·264607
1·05	5889	5889·97	0·265853	0·266418

Table 3. Critical Reynolds and wave numbers obtained by different authors

	Method	R_c	α_c
Thomas [27]	Finite difference	5780	1·026
Orszag [12]	Chebyshev polynomial expansion	5772·22	1·02056
Chock and Schechter [13]	Runge–Kutta	5772·225	1·0205
Platten [3]	Local potential variational technique	$5600 < R_c < 5900$	$\simeq 1$

Values proposed by other authors are presented in Table 3. The comparison with our values reflects a very good agreement. In particular, it appears that our results are better than those of Platten, although in both cases the self-consistent method was used. However, this is not surprising because both of the analyses differ by the expressions of the functionals I, by the choice of the trial functions and by the numerical methods to determine the eigenvalues.

5. THE PLANE COUETTE FLOW WITH TEMPERATURE DEPENDENT VISCOSITY

The fluid moves between two plates in relative motion with a constant velocity v^*. Both plates are at the same temperature. The density and the heat conductivity of the fluid are constant while the viscosity depends on the temperature according to the exponential law (3.4).

The solutions of the basic stationary flow equations are [28]:

$$\theta = \beta + \ln(a\,\mathrm{sech}^2\, bY), \tag{5.1}$$

$$U = \tfrac{1}{2}(1 + e\tanh bY), \tag{5.2}$$

where

$$a = 1 + \frac{B_r}{8}, \quad b = \sinh^{-1}\left(\frac{B_r}{8}\right)^{1/2},$$

$$e = \left(\frac{1 + B_r/8}{B_r/8}\right)^{1/2}. \tag{5.3}$$

The momentum balance equation yields

$$gDU = \zeta = \text{constant}. \tag{5.4}$$

Moreover the maximum shear stress which can be applied to the walls corresponds to a Brinkmann number equal to 18·152 [28].

The variational equation to be considered is given by the general expression (3.20) wherein the last term must be dropped according to (5.4).

For the trial functions W and Θ, we choose linear combination of the Chebyshev polynomials T_i. In order to satisfy the boundary conditions (3.26), we take:

$$W = \sum_{i=1}^{n} a_i f_i(Y) + \sum_{i=1}^{n} b_i g_i(Y) \tag{5.5}$$

$$\Theta = \sum_{i=1}^{n} d_i p_i(Y) + \sum_{i=1}^{n} e_i q_i(Y) \tag{5.6}$$

and likewise for W_0 and Θ_0, with

$$f_i = T_{2i+3} - \left(\frac{i^2+3i}{2} + 1\right)T_3 + \frac{i^2+3i}{2}T_1, \tag{5.7}$$

$$g_i = T_{2i+2} - (i+1)^2 T_2 + [(i+1)^2 - 1]T_0, \tag{5.8}$$

$$p_i = T_{2i+1} - T_1, \qquad q_i = T_{2i} - T_0; \tag{5.9}$$

it is easy to verify that

$$f_i(\pm 1) = Df_i(\pm 1) = g_i(\pm 1) = Dg_i(\pm 1) = 0$$

and

$$p_i(\pm 1) = q_i(\pm 1) = 0.$$

When Chandrasekhar–Reid functions are chosen for the velocity disturbance, the numerical procedure is unstable for values of B_r smaller than 19, i.e. precisely in the physical region of interest. Therefore, we have taken Chebyshev polynomials. Moreover, for uniformity reasons, we have selected the same functions for the temperature disturbance.

Introducing (5.5) and (5.6) in the expression (3.20) of the functional I and applying the self-consistent method, we obtain the characteristic matrix,

$$\begin{bmatrix} -i(\mathbf{A}^2)^{-1}\mathbf{A}^1 & (\mathbf{A}^2)^{-1}\mathbf{B}^1 & -i(\mathbf{A}^2)^{-1}\mathbf{C}^1 & 0 \\ -(\mathbf{B}^3)^{-1}\mathbf{A}^3 & -i(\mathbf{B}^3)^{-1}\mathbf{B}^2 & 0 & -i(\mathbf{B}^3)^{-1}\mathbf{D}^1 \\ -i(\mathbf{C}^3)^{-1}(\mathbf{C}^1)^T & (\mathbf{C}^3)^{-1}\mathbf{B}^4 & -i(\mathbf{C}^3)^{-1}\mathbf{C}^2 & (\mathbf{C}^3)^{-1}\mathbf{D}^2 \\ -(\mathbf{D}^4)^{-1}\mathbf{A}^4 & -i(\mathbf{D}^4)^{-1}(\mathbf{D}^1)^T & (\mathbf{D}^4)^{-1}(\mathbf{D}^2)^T & -i(\mathbf{D}^4)^{-1}\mathbf{D}^3 \end{bmatrix} \tag{5.10}$$

Table 4. Convergence of the imaginary part c_i of the stability parameter

	$\alpha = 1$			$P_r = 1$		
	$B_r = 1$		$B_r = 10$		$B_r = 40$	
n \ c_i	$R = 10\cdot000$	$R = 40\cdot000$	$R = 5\cdot000$	$R = 20\cdot000$	$R = 8$	$R = 9$
2	$-0\cdot00353$	$-0\cdot00088$	$+0\cdot00353$	$+0\cdot00115$	$+0\cdot05164$	$+0\cdot08727$
3	$+0\cdot00152$	$+0\cdot00030$	$+0\cdot00675$	$+0\cdot00276$	$-0\cdot09773$	$+0\cdot06057$
4	$+0\cdot00291$	$+0\cdot00364$	$-0\cdot00434$	$+0\cdot00364$	$-0\cdot04726$	$+0\cdot04927$
5	$-0\cdot00181$	$+0\cdot00462$	$-0\cdot02369$	$+0\cdot00394$	$-0\cdot02291$	$+0\cdot04928$
6	$-0\cdot00191$	$+0\cdot00048$	$-0\cdot01308$	$+0\cdot00295$	$-0\cdot02031$	$+0\cdot04947$
7	$-0\cdot00281$	$+0\cdot00761$	$-0\cdot02666$	$+0\cdot00306$	$-0\cdot02056$	$+0\cdot04950$

Table 5. Values of R and α giving neutral stability for various assigned values of B_r and P_r

	$B_r = 1$			$B_r = 5$			$B_r = 15$		
α \ R	$P_r = 1$	$P_r = 5$	$P_r = 10$	$P_r = 1$	$P_r = 5$	$P_r = 10$	$P_r = 1$	$P_r = 5$	$P_r = 10$
0·6		11310	9060	9415	6470	6085	4700	4275	4270
0·8	19530	9690	7650	7970	5270	4940	4615	4135	4070
0·9	17850	9400	7360	7665	4950	4625	4595	4130	4055
1	16580	9390	7290	7570	4750	4430	4550	3880	3805
1·1	15605	9690	7462	7680	4670	4345	4530	3720	3640
1·2	14870	10445	7975	8025	4715	4370	4490	3635	3555
1·3	14330	12015	9125	8670	4915	4535	4560	3635	3550
1·4	13955	13025	12030	9830	5360	4925	4770	3735	3645
1·6	13665		12630		9300	8470	6030	4515	4400
1·7	13760		12690						

superscript T denotes transposition, the explicit expressions of the matrices $\mathbf{A}^i$, $\mathbf{B}^i$, $\mathbf{C}^i$ and $\mathbf{D}^i$ ($i = 1, 2, 3, 4$) in terms of the functions f_i, g_i, p_i and q_i are given in the appendix. The eigenvalues of this matrix are the parameters c of the stability problem; as above, they are determined by the $Q - R$ method.

When more than seven terms are used in the expansions (5.5) and (5.6), the results lose any significance because of numerical instabilities. In Table 4 the values of c_i are reported for some values of the parameters α, P_r*, B_r and R. The convergence is not very satisfactory especially for low values of B_r. Nevertheless, there is no problem to define unambiguously the stable ($c_i < 0$) and the unstable ($c_i > 0$) region. For instance, it is clear that in the case $B_r = 10$, the system is stable for $R < 5000$ and unstable for $R > 20\cdot000$. Likewise for $B_r = 40$, it is seen that there is stability as long as $R \leqslant 8$ and instability for $R \geqslant 9$. Of course, the latter case is purely academic since the maximum shear stress corresponds to $B_r = 18\cdot152$.

Taking five terms in the developments (5.5) and (5.6) and fixing the Brinkmann and the Prandtl numbers, we have computed the Reynolds number corresponding to neutral stability for various assigned values of α (see Table 5).

We have also derived the critical values of the Reynolds and the wave numbers for $B_r = 1, 5, 10, 15$ and 40 when the Prandtl number takes the values 1, 5 and 10. Due to the lack of accuracy of the numerical procedure, these results must be interpreted as giving the order of magnitude.

* P_r is the Prandtl number; it is defined by

$$P_r = P_e R^{-1} = \frac{\eta^*}{\rho \chi^*}$$

where $\chi^* = \lambda/\rho c_p$ is the heat diffusivity.

Table 6. Critical values of R and α for various assigned values of B_r and P_r and comparison with the results of Sukanek *et al.* [21]

		Lebon–Nguyen		Sukanek *et al.*	
B_r	P_r	R_c	α_c	R_c	α_c
1	1	13665	1·60		
	5	9370	0·95		
	10	7290	1		
5	1	7570	1		
	5	4670	1·10		
	10	4340	1·15		
10	1	5160	1·10		
	5	3835	1·20		
	10	3700	1·20		
15	1	4490	1·20	3500	0·96
	5	3625	1·25	3475	0·98
	10	3540	1·25		
40	1	44	0·8	116	0·47
	5	47	0·8	68	0·65

The results are reproduced in Table 6 together with those obtained by Sukanek *et al.* [21]. These authors expanded W in terms of the Chandrasekhar–Reid functions and Θ in terms of Fourier's series; n was taken equal to four. It must however be pointed out that their expansion for Θ does not obey the boundary conditions $\Theta(\pm 1) = 0$! Moreover, their characteristic matrix with complex elements is mapped into a real one by a suitable transformation operator. Due to numerical instability, Sukanek *et al.* were not able to compute R_c and α_c for B_r lower than 15. In our work, by using Chebyshev polynomials and working with complex variables, we extended this limit to $B_r = 1$. Although our numerical values are different from those of [21], especially for $B_r = 40$, we observe the same general tendency. For a fixed Prandtl number, the critical Reynolds number increases when the Brinkmann number decreases; at $B_r = 0$ (non-viscous fluid), it is reasonable to expect that $R_c = \infty$ so that the system remains indefinitely stable. On the other hand, when the Brinkmann number is given, R increases generally when the Prandtl number decreases.

6. CONCLUSIONS

The purpose of this work was twofold: extend the variational principle of Lebon and Lambermont to stability problems and apply it to specific examples. The approximate solutions were obtained by using the self-consistent technique. Like any variational method, it is rather sensitive to the choice of the trial functions. For the isothermal Poiseuille flow, the amplitude of the disturbed velocity was expressed in terms of the Chandrasekhar–Reid functions; the solutions obtained converge quite well and are in fair agreement with those given by other authors. In the case of the Couette flow with temperature dependent viscosity, the disturbed velocity and temperature were expanded in terms of the Chebyshev polynomials. However, because of lack of convergence of the solutions, the neutral stability curves and the critical values of the parameters could not be determined with great precision. Therefore, concerning the Couette flow, our results must be considered as a first estimate. Nevertheless, in our opinion, our contribution may be useful in that it gives a general pattern for the flow and constitutes a first step for further studies of the problem.

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APPENDIX

$$Q^1_{ij} = \int_{-1}^{+1} ch^2bY(D^2f_i)(D^2f_j)\,\mathrm{d}Y$$

$$Q^2_{ij} = \int_{-1}^{+1} ch^2bY(Df_i)(Df_j)\,\mathrm{d}Y$$

$$Q^3_{ij} = \int_{-1}^{+1} ch^2bYf_if_j\,\mathrm{d}Y$$

$$Q^4_{ij} = \int_{-1}^{+1} (Df_i)(Df_j)\,\mathrm{d}Y$$

$$Q^5_{ij} = \tfrac{1}{2}\int_{-1}^{+1} f_if_j\,\mathrm{d}Y$$

$$P^1_{ij} = \int_{-1}^{+1} \frac{th(by)}{ch^2bY} f_ig_j\,\mathrm{d}Y$$

$$P^2_{ij} = \int_{-1}^{+1} th(by)f_i(D^2g_j)\,\mathrm{d}Y$$

$$P^3_{ij} = \int_{-1}^{+1} th(by)f_ig_j\,\mathrm{d}Y$$

$$P^4_{ij} = \int_{-1}^{+1} th(by)g_iD^2f_j\,\mathrm{d}Y$$

$$L^1_{ij} = \int_{-1}^{+1} (Df_i)(Dp_j)\,\mathrm{d}Y$$

$$L^2_{ij} = \int_{-1}^{+1} f_ip_j\,\mathrm{d}Y$$

$$M^1_{ij} = \int_{-1}^{+1} ch^2bY(D^2g_i)(D^2g_j)\,\mathrm{d}Y$$

$$M^2_{ij} = \int_{-1}^{+1} ch^2bY(Dg_i)(Dg_j)\,\mathrm{d}Y$$

$$M^3_{ij} = \int_{-1}^{+1} ch^2bYg_ig_j\,\mathrm{d}Y$$

$$M^4_{ij} = \int_{-1}^{+1} (Dg_i)(Dg_j)\,\mathrm{d}Y$$

$$M^5_{ij} = \tfrac{1}{2}\int_{-1}^{+1} g_ig_j\,\mathrm{d}Y$$

$$N^1_{ij} = \int_{-1}^{+1} (Dg_i)(Dq_j)\,\mathrm{d}Y$$

$$N^2_{ij} = \int_{-1}^{+1} g_iq_j\,\mathrm{d}Y$$

$$S^1_{ij} = \int_{-1}^{+1} th(by)p_ig_j\,\mathrm{d}Y$$

$$O^1_{ij} = \int_{-1}^{+1} (Dp_i)(Dp_j)\,\mathrm{d}Y$$

$$O^2_{ij} = \int_{-1}^{+1} \frac{1}{ch^2bY} p_ip_j\,\mathrm{d}Y$$

$$O^3_{ij} = \int_{-1}^{+1} p_ip_j\,\mathrm{d}Y$$

$$R^1_{ij} = \int_{-1}^{+1} th(by)p_iq_j\,\mathrm{d}Y$$

$$U^1_{ij} = \int_{-1}^{+1} th(by)q_if_j\,\mathrm{d}Y$$

$$T^1_{ij} = \int_{-1}^{+1} (Dq_i)(Dq_j)\,\mathrm{d}Y$$

$$T^2_{ij} = \int_{-1}^{+1} \frac{q_iq_j}{ch^2bY}$$

$$T^3_{ij} = \int_{-1}^{+1} q_iq_j\,\mathrm{d}Y$$

$$A^1_{ij} = \frac{2B_r}{a}(Q^1_{ij} + 2\alpha^2Q^2_{ij} + \alpha^4Q^3_{ij} + 4b^2\alpha^2(Q^3_{ij} - Q^5_{ij}))$$

$$A^2_{ij} = \alpha RB_r(Q^4_{ij} + 2\alpha^2Q^5_{ij})$$

$$B^1_{ij} = 2eB_r\alpha R\left(\frac{\alpha^2}{2}P^3_{ij} - b^2P^1_{ij} - \tfrac{1}{2}P^2_{ij}\right)$$

$$B^2_{ij} = \frac{2B_r}{a}(M^1_{ij} + 2\alpha^2M^2_{ij} + \alpha^4M^3_{ij} + 4b^2\alpha^2(M^3_{ij} - M^5_{ij}))$$

$$B^3_{ij} = \alpha RB_r(M^4_{ij} + 2\alpha^2M^5_{ij})$$

$$B^4_{ij} = -2\alpha bP_eS^1_{ij}$$

$$A^3_{ij} = 2\alpha ReB_r\left(-\frac{\alpha^2}{2}(P^3_{ij})^T + b^2(P^1_{ij})^T + \tfrac{1}{2}P^4_{ij}\right)$$

$$A^4_{ij} = 2b\alpha P_eU^1_{ij}$$

$$C^1_{ij} = \frac{eb}{a}B_r(\alpha^2L^2_{ij} - L^1_{ij})$$

$$C^2_{ij} = O^1_{ij} + \alpha^2O^3_{ij} + \frac{b^2e^2}{4a}B_rO^2_{ij}$$

$$C_{ij}^3 = \frac{\alpha P_e}{2} O_{ij}^3 \qquad D_{ij}^3 = T^1 + \alpha^2 T_{ij}^2 + \frac{b^2 e^2}{4a} B_r T_{ij}^2$$

$$D_{ij}^1 = \frac{eb}{a} B_r \{-N_{ij}^1 + \alpha^2 N_{ij}^2\} \qquad D_{ij}^4 = \frac{\alpha P_e}{2} T_{ij}^3.$$

$$D_{ij}^2 = \frac{\alpha P_e}{2} e R_{ij}^1$$

STABILITE HYDRODYNAMIQUE PAR METHODES VARIATIONNELLES

Résumé—On propose un principe variationnel général pour l'étude de la stabilité linéaire des écoulements stationnaires et non-isotherme. On montre que les équations d'Euler–Lagrange du critère variationnel sont les relations d'Orr–Sommerfeld du problème de stabilité. La théorie générale est appliquée à deux exemples. On considère tout d'abord l'écoulement isotherme de Poiseuille entre deux plans parallèles infinis: en utilisant la méthode "self-consistent" de Glansdorff–Prigognine; on trouve un nombre de Reynolds critique du même order de grandeur que celui obtenu par d'autres auteurs. Comme second exemple, on étudie la conséquence d'une viscosité fonction de la température sur la stabilité d'un écoulement de Couette plan. On montre que pour certaines valeurs de paramètres, l'écoulement devient instable. Nos résultats sont comparés avec ceux obtenus par Sukanek *et al.* qui furent, à notre connaissance, les premiers à traiter ce problème.

EINE STUDIE ÜBER HYDRODYNAMISCHE STABILITÄT BEI VARIATIONSMETHODEN

Zusammenfassung—Es wird eine allgemeine Variationsgleichung für die Untersuchung der linearen Stabilität von nichtisothermen Flüssigkeitsströmungen vorgeschlagen. Es wird gezeigt, daß die Euler-Lagrange-Gleichungen des Variationskriteriums die Orr-Sommerfeld-Beziehungen des Stabilitätsproblems sind. Die allgemeine Theorie wurde auf zwei spezifische Beispiele angewandt. Erstens betrachtete man die isotherme Poiseuille-Strömung zwischen zwei unendlichen parallelen Platten. Man wandte die selbstkonsistente Technik an, die von Glansdorff und Prigogine eingeführt wurde und fand heraus, daß die kritische Reynolds-Zahl von der gleichen Größenordnung ist wie Werte anderer Autoren. Als zweites Beispiel untersuchte man den Einfluß einer temperaturabhängigen Viskosität auf die Stabilität einer ebenen Couetteströmung. Es wird gezeigt, daß für gewisse Parameterwerte die Strömung instabil wird. Unsere Ergebnisse wurden mit denen von Sukanek verglichen, die unseres Wissens die ersten sind, die dieses Problem behandeln.

ИССЛЕДОВАНИЕ ГИДРОДИНАМИЧЕСКОЙ УСТОЙЧИВОСТИ ВАРИАЦИОННЫМИ МЕТОДАМИ

Аннотация — Предложено общее вариационное уравнение для изучения линейной устойчивости неизотермических течений жидкости. Показано, что уравнения Эйлера-Лагранжа вариационного критерия представляют собой соотношения Орра-Зоммерфельда в задаче об устойчивости. Общая теория применяется к двум частным случаям. В первом случае рассматривается изотермическое течение Пуазейля между двумя бесконечными параллельными плоскостями. С помощью метода самосогласования, предложенного Кланcдорфом и Пригожиным, найдено, что критическое число Рейнольдса имеет тот же порядок величины, что и его значения, полученные другими авторами. Во втором случае исследуется влияние вязкости, зависящей от температуры, на устойчивость плоского течения Куэтта. Показано, что при некоторых значениях параметров течение становится неустойчивым. Наши результаты сравниваются с данными Суканека и других, которые, по нашему мнению, являются первыми опубликованными данными по этой проблеме.